Entered

Honors Thesis Proposal

For

Gallai-Ramsey Numbers for Cycles of Length Seven

Dylan Bruce

Zixia Song, Ph.D.
Thesis Committee Chair
College of Sciences

Xin Li, Ph.D.
Department Chair, Department of Mathematics
College of Sciences

Joseph Brennan, Ph.D.
Committee Member from Major
College of Sciences

Gallai-Ramsey Numbers for Cycles of Length Seven

Dylan Bruce

December 8, 2016

1 Introduction

To understand some developments of modern Ramsey theory, we must first explain the meaning behind the terminology. A graph G is a pair of a set of vertices (denoted by $V(G)$) and a set of edges (denoted by $E(G)$), where each edge connects two different vertices (the edge set can be considered a set of 2 element subsets of the vertex set). An endpoint of an edge is one of the two vertices that an edge connects, and an edge is incident to a vertex v if v is one of the endpoints. An edge-coloring on the graph G is a function ϕ which assigns each element of $E(G)$ a natural number, where the natural number assigned is the edge's color. A complete graph on n vertices (denoted by K_n) is a graph on n vertices such that there exists an edge between each pair of vertices. A subgraph H of a graph G is a pair of a subset of the vertices of G and a subset of the edges incident to those vertices. A cycle is a sequence of vertices such that each vertex is connected to the previous and the next vertex in the sequence, and the only repeated vertex is the beginning and end vertex. A path is a sequence of vertices such that there exists an edge connecting each pair of successive vertices, where each vertex in the sequence is unique.

Ramsey theory has its origins from the work of Frank Ramsey, publishing a paper in 1930 in which he proved the following theorem. [1]

Theorem 1. *For any given integer c, any given integers $n_1, ..., n_c$, there is a number, $R(n_1, ..., n_c)$, such that if the edges of a complete graph of order $R(n_1, ..., n_c)$ are colored with c different colors, then for some i between 1 and c, it must contain a complete subgraph of order n_i whose edges are all color i.*

In general, we define the Ramsey number of $n_1, ..., n_c$ as the minimum number of vertices n such that a complete graph on n vertices will contain a complete subgraph of n_1 vertices colored color 1, or a complete subgraph of n_2 vertices colored color 2, and so forth.

We observe a special case of Ramsey's theorem where $c = 2$, such that the graph's edges are colored by two colors (we generally use red and blue to represent these two colors), wherein we will find either a complete subgraph of n_1 vertices which is edge colored only by red, or a complete subgraph of n_2 vertices which is edge colored only by blue. We are able to calculate exact values for $R(n_1, ..., n_c)$ in two steps: showing that for every edge-coloring of a K_n we find a K_{n_1} colored with color 1, or a K_{n_2} colored with color 2, and so forth, and then by showing the existence of a coloring of K_{n-1} which has no K_{n_1} colored with color 1, or K_{n_2} colored with color 2, and so forth. A common introductory example is to show that $R(3, 3) = 6$.

However, further developments in Ramsey theory do not restrict themselves to only finding monochromatic complete subgraphs. Instead of looking for complete subgraphs, we may also look for any other family of graphs. Other families of graphs include paths or cycles. Additionally, instead of looking for monochromatic subgraphs, we may instead concern ourselves with graphs that are known as rainbow, which are edge colorings such that each edge is colored with a different color. Searching for subgraphs that are either a rainbow copy of one graph, or a monochromatic copy of another graph leads us to what is known as Gallai-Ramsey theory.

2 Gallai-Ramsey Numbers

Similar to Ramsey numbers, the Gallai-Ramsey number of a graph is related to the minimum number of vertices a complete graph must contain such that when the graph is edge colored by some number of colors, the graph will always contain a certain monochromatic graph, or a certain rainbow graph. In terms of mathematical notation, we denote the Gallai-Ramsey number as $\mathrm{gr}_k(G : H)$, where a complete graph of $\mathrm{gr}_k(G : H)$ vertices edge colored by k colors will always contain either a rainbow subgraph G or a monochromatic subgraph H. Gallai-Ramsey theory has its origins in a paper published in 1967 by Gallai, in which he proved the following result. [2]

Theorem 2. *In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition (a partition into more than one part) of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

Due to this result, we define rainbow triangle free colorings as Gallai colorings. This theorem has proven to be quite useful in proving results for Gallai-Ramsey numbers where we consider the case of rainbow triangles. In our research, we will be studying the Gallai-Ramsey number of Gallai colorings of graphs.

In their recent paper, Hall, Magnant, Ozeki, and Tsugaki presented strengthened upper bounds on the Gallai-Ramsey numbers when searching for monochromatic paths and cycles. [3] In addition, Faudree, Gould, Jacobson, and Magnant presented lower bounds for the Gallai-Ramsey numbers for monochromatic paths, [4] and Fujita and Magnant presented lower bounds for the Gallai-Ramsey numbers for monochromatic cycles. [5] These results combined give the following best known bounds.

Theorem 3. *For all integers k and n with $k \geq 1$ and $n \geq 2$,*

$$(n-1)k + n + 1 \leq gr_k(K_3 : C_{2n}) \leq (n-1)k + 3n$$

$$n2^k + 1 \leq gr_k(K_3 : C_{2n+1}) \leq (2^{k+1} - 3)n \log n.$$

For all integers k and n with $k \geq 1$ and $n \geq 3$,

$$\left\lfloor \frac{n-2}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil + 1 \leq gr_k(K_3 : P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor k + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

Using these formulas, we can provide a range for the Gallai-Ramsey numbers of paths and cycles. Of particular note, we find that for small odd cycles and k number of colors, we have the bounds

$$2^k + 1 \leq \mathrm{gr}_k(K_3 : C_7) \leq (2^{k+1} - 3)3 \log 3.$$

3 Exact Gallai-Ramsey Numbers for Small Odd Cycles

Although we have formulas to determine bounds on the Gallai-Ramsey number for the general cases, the precise Gallai-Ramsey number, even for the simpler cases, is difficult to compute. As an example, we will consider the case of C_3. To show the lower bounds of any kind of Ramsey number, we can exhibit an edge coloring on a complete graph of $n-1$ vertices that does not satisfy any of the desired conditions; for example, the following coloring with 3 colors on a complete graph of 10 vertices contains no monochromatic C_3 and no rainbow K_3. Thus, $\mathrm{gr}_3(K_3 : C_3) > 10$.

In fact, the coloring of Figure 1 can be derived by using Theorem 2. We can do so in the following way: label the bottom vertex of a K_{10} 1, and sequentially label each successive vertex clockwise the next natural number up to 10. Place the vertices with odd label into one partition, and the vertices with even label into another. Then each partition has 5 vertices, and as stated before, since $R(3,3) = 6$, we can find a 2 coloring of the edges of each partition which has no monochromatic triangle. Then, color every edge connecting the two partitions the third color (define this color blue).

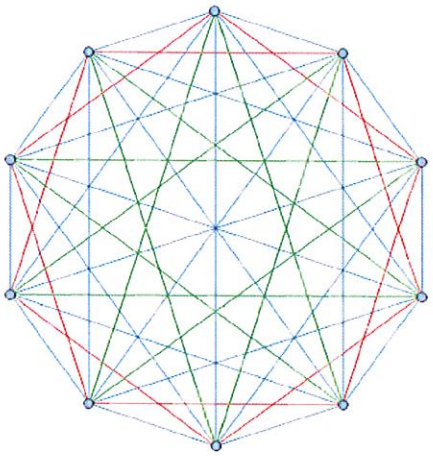

Figure 1: A 3 edge coloring of K_{10} which exhibits no monochromatic or rainbow triangle

By theorem 2, since this partition exists, this coloring is rainbow triangle free, and since the subgraph of K_{10} induced by blue is a bipartite graph, it contains no odd cycles, thus this coloring is also monochromatic triangle free. So this coloring demonstrates that $\mathrm{gr}_3(K_3 : C_3) > 10$.

In fact, it turns out that $\mathrm{gr}_3(K_3 : C_3) = 11$. This is a result due to Chung and Graham, who proved the following general formula. [6]

Theorem 4.

$$gr_k(K_3 : C_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even} \\ 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd} \end{cases}$$

4 Current Research

In our research, we wish to further develop our knowledge of exact Gallai-Ramsey numbers for small odd cycles. The case for a cycle of 3 vertices has already been settled, as shown above by Theorem 4. However, we wish to find the precise values for further cases, in particular, we would like to provide a general formula for the Gallai-Ramsey number for cycles of length 7 for edge colorings of k colors. This has proven to be a nontrivial task, with the only known results being the general bounds outlined in Theorem 3. Given those bounds, we find that, for instance, when colored by three colors, we find that

$$9 \leq \mathrm{gr}_3(K_3 : C_7) \leq 39 \log 3 \leq 43.$$

This provides us with a starting point for one choice of the number of colors. In order to show the Gallai-Ramsey number for each case of k colors and an odd cycle C_{2r+1} is exactly n, we need to proceed through two steps: first, we need to show that every k edge coloring of a K_n contains either a monochromatic C_{2r+1} or a rainbow triangle. Second, we need to show that there exists an edge coloring ϕ of K_{n-1} such that there exists neither a monochromatic C_{2r+1} nor a rainbow triangle. Completing the first step demonstrates that $\mathrm{gr}_k(K_3 : C_{2r+1}) \leq n$, since the Ramsey number is the minimum number of vertices we need to always be able to find certain subgraphs. Completing the second step demonstrates that $\mathrm{gr}_k(K_3 : C_{2r+1}) > n-1$, since exhibiting a coloring without the desired subgraphs clearly means that not every edge coloring of K_{n-1} satisfies the conditions, so the minimum number of vertices needed is larger than $n-1$. Thus, these inequalities combined will give us that $\mathrm{gr}_k(K_3 : C_{2r+1}) = n$. Although the proof structure is easy to understand, providing the proofs is anything but trivial, as evidenced by the fact that the only exact numbers known are those for cycles up to length 6. Providing exact Gallai-Ramsey numbers for cycles of length 7 would thus be a significant development.

References

[1] F. P. Ramsey, On a problem of formal logic, Proceedings of the London Mathematical Society 30 (1930), 264-286.

[2] T. Gallai, Transitiv orientierbare Graphen, Acta Math Acad Sci Hungar 18 (1967), 25-67

[3] M. Hall, C. Magnant, K. Ozeki, M. Tsugaki, Improved Upper Bounds for Gallai-Ramsey Numbers of Paths and Cycles, J. Graph Theory 75(1) (2014), 59 - 74

[4] R. J. Faudree, R. Gould, M. Jacobson, and C. Magnant, Ramsey numbers in rainbow triangle free colorings, Australas J Combin, 46 (2010), 269 - 284.

[5] S. Fujita and C. Magnant, Gallai-Ramsey numbers for cycles, Discrete Math 311(13) (2011), 1247 - 1254.

[6] F. R. K. Chung and R. Graham, Edge-colored complete graphs with precisely colored subgraphs, Combinatorica 3 (1983), 315324.